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Optimal control of maize foliar diseases using the plants population dynamics

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Pathogens and insects can have important negative effects on yields of crops cultivated by humans. These effects can be important for the food security or financial well-being of individuals. In particular, maize is a very important staple crop worldwide and is vulnerable to diseases. We formulate here a mathematical model to evaluate the impacts of foliar diseases on the population dynamics of maize plants. Qualitative analyses of the important mathematical features of the model are carried out. We show how this methodology can be extended to reducing the spread of foliar diseases through effective control measures with minimum costs.

Keywords: maize foliar diseases; optimal control; population dynamics; stability analysis.

Introduction

Modern humans predominantly settle permanently due mostly to the development of agricultural practices. Pathogens and insects have evolved along-side plants and are viewed as pests when they reduce yields, and so controlling such pests is a practice as old as agriculture (Kingsland 1980). There is increasing appreciation of the crucial role that pathogens may play in the structure of plant populations and communities (Vandermeer 1990). The use of models for understanding the development of epidemics has developed from the 1960s into an important methodology in understanding plant diseases (Van Maanen & Xu 2003).

Maize (*Zea mays L.* ssp. *mays*), grown commercially or for consumption, is an important crop worldwide. Many diseases can attack maize through the leaves of the plant. Generally, epidemics occur as the result of interactions among three factors, the plant population dynamics, the population dynamics of the pathogens and impacts from the environment. These basic factors allow the development and use of population dynamic models to assess, predict and/or control the epidemic.

Model development

To improve our understanding of the impact of foliar diseases on the dynamics of maize plant populations and to give some indication of how to reduce the spread of the diseases with minimum cost, we formulate a mathematical epidemiological model. We partition the maize plant population into two categories: a disease-free population of susceptible maize plants S and a population of infected maize plants I. P is a population measure of foliar disease pathogens. Based on these assumptions, we construct the model

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda - \beta S(t)P(t) - \mu_1 S(t),\\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \beta S(t)P(t) - \mu_2 I(t), \qquad (1)\\
\frac{\mathrm{d}P}{\mathrm{d}t} &= \sigma I(t) - (d-b)P(t),
\end{aligned}$$

with initial conditions: $S(0) > 0,\ I(0) \geq 0,\ P(0) \geq 0$. For simplicity, we let $\xi = d - b > 0$ be the net decay rate of $P(t)$. The meaning of variables and parameters are given in Table 1. Note that the total

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Table 1. Variables and parameters for model (1).

Variables	Meaning	Unit
$N(t)$	Total number of maize plants	plants ha^{-1}
$S(t)$	Susceptible maize plants	plants ha^{-1}
$I(t)$	Infected maize plants	plants ha^{-1}
$P(t)$	Measure of foliar disease pathogens	cells ha^{-1}
Λ	Recruitment rate into $S(t)$	plants ha^{-1} day^{-1}
β	Contact rate of maize plants with $P(t)$	ha cells^{-1} day^{-1}
ξ	Net decay rate of pathogens	day^{-1}
σ	Contributions of $I(t)$ to the growth of $P(t)$	cells plants^{-1} day^{-1}
μ_1	Death rate of $S(t)$	day^{-1}
μ_2	Death rate of $I(t)$ due to foliar pathogens	day^{-1}

maize plant population at any time t is given by $N(t) = S(t) + I(t)$.

Model analysis

The foliar disease-free equilibrium (DFE) point of model (1) is given by

$$(S^0, I^0, P^0) = (\Lambda/\mu_1, 0, 0). \tag{2}$$

Threshold quantity (the basic reproduction number R_0)

To analyze the stability of this equilibrium point, we consider a threshold quantity known as the basic reproduction number. The basic reproduction number measures the expected number of secondary infections that result from a single infected maize plant in a maize plant population made up entirely of susceptible plants. We compute the basic reproduction number R_0 of model (1) using the next-generation matrix approach of van den Driessche and Watmough (2002), given by

$$R_0 = \frac{\beta\sigma\Lambda}{\mu_1\xi\mu_2}. \tag{3}$$

Stability analysis of the DFE

The stability of the DFE determines the short-term dynamics of a disease (Liao and Wang 2011). Therefore, to determine the short-term dynamics of maize foliar disease, it is necessary to investigate the stability of the DFE.

THEOREM 1 *The DFE of model (1) is locally asymptotically stable provided that $R_0 < 1$.*

Proof

We show that all the eigenvalues of the Jocobian of model (1) evaluated at the DFE have negative real parts. The Jacobian matrix $\mathcal{J}^0$ of model (1) evaluated at the DFE is

$$\mathcal{J}^0 = \begin{pmatrix} \mu_1 & 0 & -\beta S^0 & \\ 0 & -\mu_2 & \beta S^0 & \\ 0 & \sigma & -\xi & 0 \end{pmatrix}. \tag{4}$$

The Jacobian matrix $\mathcal{J}^0$ has three distinct eigenvalues given by

$$\lambda_1 = -\mu_1,$$
$$\lambda_2 = \frac{1}{2}\left[-(\mu_2+\xi) - \sqrt{(\mu_2+\xi)^2 + 4\xi\mu_2(R_0-1)}\right],$$
$$\lambda_3 = \frac{1}{2}\left[-(\mu_2+\xi) + \sqrt{(\mu_2+\xi)^2 + 4\xi\mu_2(R_0-1)}\right].$$

It is obvious that all these eigenvalues are negative if $R_0 < 1$, thus completing the proof.■

This result implies that foliar diseases can be eliminated if the initial size of the infected plants is in the basin of attraction of the DFE. On the other hand, the disease will be established if $R_0 > 1$. To ensure that disease elimination is independent of the initial population size of infected plants, it is necessary to show that the DFE is globally stable. The proof of global stability will be established using a global stability result by Castillo-Chavez et al. (2002) which is stated in Lemma 2.

LEMMA 2. *Consider a model system written in the form*

$$\begin{aligned} \frac{dX_1}{dt} &= F(X_1, X_2), \\ \frac{dX_2}{dt} &= G(X_1, X_2), \quad G(X_1, 0) = 0, \end{aligned} \tag{5}$$

where $X_1 \in R^m$ and $X_2 \in R^n$. $X_0 = (X_1^*, 0)$ denotes the disease-free equilibrium of the system. Assume that

(H1). For $dX_1/dt = F(X_1, 0)$, X_1^* is globally asymptotically stable.

(H2). $G(X_1, X_2) = AX_2 - \hat{G}(X_1, X_2)$, $\hat{G}(X_1, X_2) \geq 0$ for $(X_1, X_2) \in \Omega$, where the Jacobian $A = (\partial G/\partial X_2)(X_1, 0)$ is an M-matrix (the off diagonal elements of A are non-negative) and Ω is the region where the model makes biological sense.

Then, the DFE X_0 is globally asymptotically stable provided that $R_0 < 1$ (Castillo-Chavez et al. 2002).

THEOREM 3 *The DFE of model (1) is globally asymptotically stable provided that $R_0 < 1$.*

Proof We only need to show that the conditions (H1) and (H2) hold when $R_0 < 1$. In our model (1), we have $X_1 = S,\ X_2 = (I, P)$. The system

$$\frac{dX_1}{dt} = F(X_1, 0) = \Lambda - \mu_1 S(t) \tag{6}$$

is linear and its solution can be easily found as $S(t) = S^0 + (S(0) - S^0)e^{-\mu_1 t}$. Clearly, $S(t) \to S^0$ as $t \to \infty$ regardless of the values of the initial condition. Thus, X_1^0 is globally asymptotically stable. Meanwhile

$$\frac{dX_2}{dt} = G(X_1, X_2) = \begin{pmatrix} \beta S(t)P(t) - \mu_2 I(t) \\ \sigma I(t) - \xi P(t) \end{pmatrix}. \tag{7}$$

We obtain

$$A = \begin{pmatrix} -\mu_2 & \beta S^0 \\ \sigma & -\xi \end{pmatrix}, \tag{8}$$

which is obviously an M-matrix with non-negatives off the diagonal elements. Thus, we find

$$\hat{G}(X_1, X_2) = \begin{pmatrix} \beta P(S^0 - S) \\ 0 \end{pmatrix}. \tag{9}$$

However, $\hat{G}(X_1, X_2) \geq 0$, since $S^0 \geq S$. This completes the proof. Biologically, this implies that foliar disease can be eliminated irrespective of the initial population of infected maize plants provided $R_0 < 1$.■

Endemic dynamics of foliar diseases

When $R_0 > 1$, a unique endemic equilibrium (EE) exists for model (1), that is an equilibrium that includes diseased populations, and is given by

$$(S^e,\ I^e,\ P^e) = \left(\frac{\xi\mu_2}{\sigma\beta}, \frac{\Lambda - \mu_1 S^e}{\mu_2}, \frac{\sigma I^e}{\xi}\right). \tag{10}$$

The stability analysis of this EE describes the long-term dynamics of the maize plant population dynamics. However, the analytical analysis necessary is complex and so we investigate the EE numerically. For the numerical simulations, we consider theoretical field trials with spacings of 0.9 m between the rows and 0.3 within the rows. Two seeds are planted per station and thinned down to one plant at three weeks after seedling emergence. So, each row has a maximum potential resulting in a total population of 44,000 plants per ha. The trials are rain-fed and harvest is done at a physiological maturity of 151 days after planting. Most of the parameters are extracted from published data while the remaining are estimated such that they lie within realistic parameter ranges. The recruitment rate Λ is estimated as the product of the germination rate and the initial number of maize plants. Germination rate = 0.95 is extracted from corn newsletters (Newsletters 2015). According to the newsletters, the survival rates for corn is in the range of 85% to 95% but can vary considerably depending on planting conditions and other environmental factors (Newsletters 2015). Based on these, the death rates will lie in the range of 5–15%. So, we take the death rate $\mu_1 = 0.5/151$ and $\mu_2 = 0.15/151$, where 151 is the number of days to maturity. By a similar approach, we estimated β from Fajinmi et al. (2012). These parameter values together with other parameter values can be found in Table 2. Numerical solutions of model (1) when $R_0 > 1$ using the parameter values are given in Figure 1.

Table 2. Parameter values.

Variables	Value	Source
Λ	276.8212	estimate
β	0.0024	Fajinmi et al. (2012)
ξ	0.85	estimate
σ	0.0018	estimate
μ_1	3.3113×10^{-4}	Newsletters (2015)
μ_2	9.9338×10^{-4}	Newsletters (2015)

Outbreak growth rate

If $R_0 > 1$, then the DFE becomes unstable and a disease outbreak is likely to occur in the maize plant population. The positive (dominant) eigenvalue of

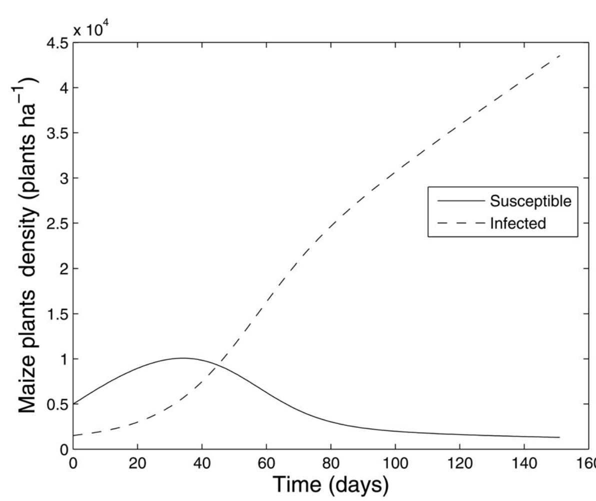

Figure 1. Graphical representation of the maize plants population dynamics in the absence of any control measure.

the Jacobian at the DFE is referred to as the initial outbreak growth rate (Tien & Earn 2010). From the eigenvalues of the Jocobian $\mathcal{J}^0$, we can see that λ_1 and λ_2 are negative. Thus, the positive (dominant) eigenvalue is given by

$$\lambda^+ = \lambda_3. \tag{11}$$

The epidemiological implications of this are that when there is no control measure to reduce the spread of an infection with $R_0 > 1$, then an outbreak will occur in the entire population and will grow at a rate λ^+. Note that if $R_0 < 1$, then all the three eigenvalues become negative confirming the result in Theorem 1. We also notice that if $R_0 = 1$, the outbreak growth rate λ^+ vanishes.

Figure 1 gives a graphical representation of the maize plant population dynamics in the absence of any control measure. From the figure, infected maize plants are increasing while the disease-free maize plants are decreasing. Thus, it is necessary to consider introducing control measures that can reduce the spread of foliar disease.

The control model

We consider an Integrated Control Management Practices. These management practices incorporate many practical methods of disease control. These measures include chemical control, biological control, plant resistance, preventive control and cultivation control. For the model, we assume that using disease resistance maize seeds reduces susceptibility to disease at a rate ϕ and with efficacy ε. Similarly, traditional practices reduces P at a rate θ and I at a rate τ. Based on these assumptions, we obtain the control model below:

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda - \beta S(t)P(t) - \mu_1 S(t) - \phi S(t) + \tau I(t),\\
\frac{\mathrm{d}C}{\mathrm{d}t} &= \phi S(t) - (1-\varepsilon)\beta C(t)P(t) - \mu_3 C(t),\\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \beta S(t)P(t) + (1-\varepsilon)\beta C(t)P(t) - (\mu_2 + \tau)I(t),\\
\frac{\mathrm{d}P}{\mathrm{d}t} &= \sigma I(t) - (\xi + \theta)P(t).
\end{aligned} \tag{12}$$

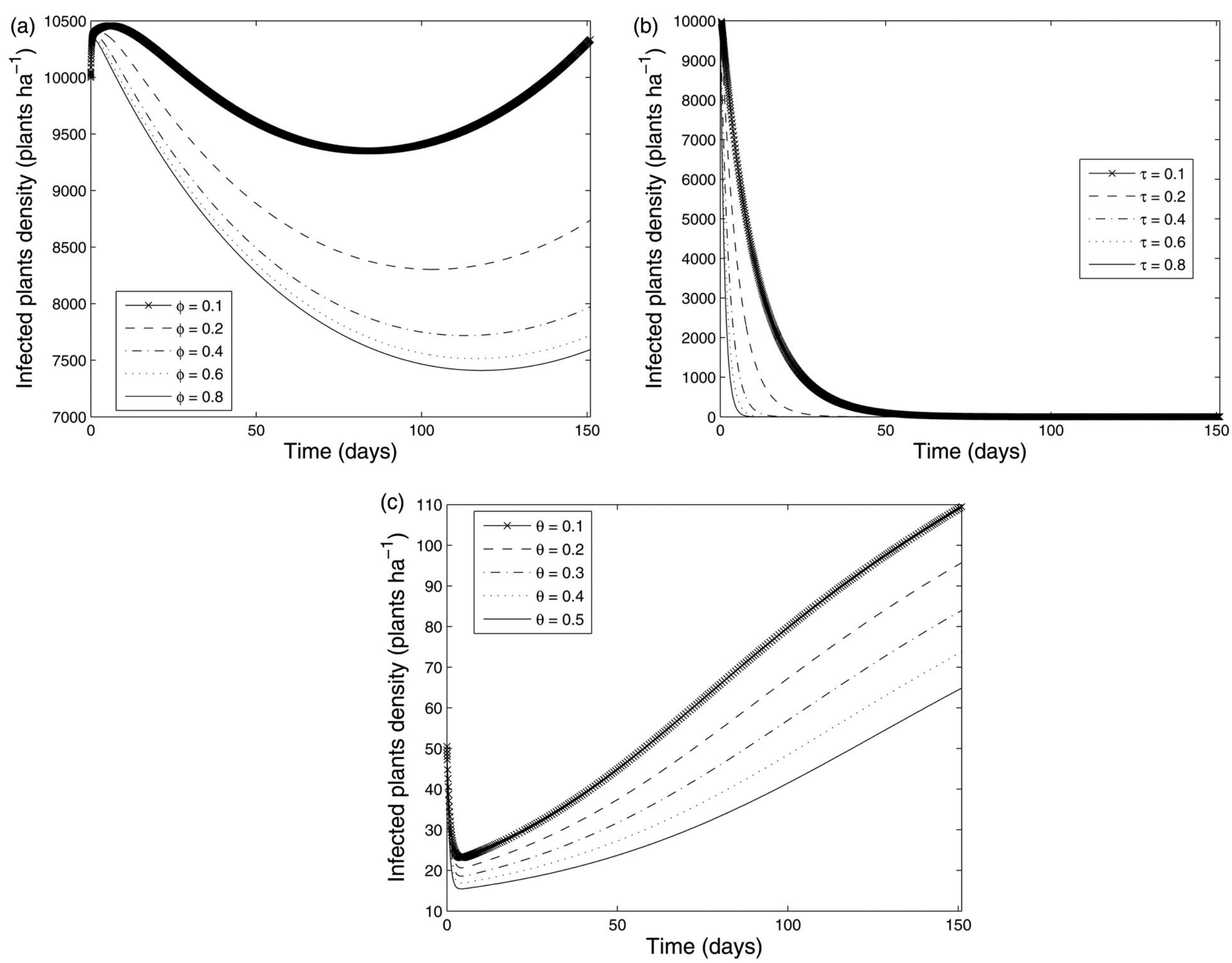

Figure 2. Plot showing effects of control measures on the plants population dynamics.

The DFE of the control model is given by

$$(S_c^0,\ C_c^0,\ I_c^0,\ P_c^0) = (\Lambda/(\mu_1+\phi),\ \phi S_c^0/\mu_3,\ 0,\ 0). \tag{13}$$

The basic reproduction number R_0^c of the control model is given by

$$R_0^c = \frac{\beta\sigma(S_c^0 + (1-\varepsilon)C_c^0)}{(\theta+\xi)(\mu_2+\tau)}. \tag{14}$$

Clearly, $R_0^c < R_0$. This shows that based on our model formulations, introduction of controls has the capacity of reducing the number of secondary infections. The relative impact of each of the control measures is determined using numerical simulation with the parameter values in Table 2 and the following parameter values for the control measures: $\tau = 10 * \mu_2,\ \varepsilon = 0.78,\ \phi = 0.75,\ \theta = 5 * \xi$. The simulations are presented in Figure 2(a)–(c). Based on our formulation, the figures reveal that each of the control measures has significant effects in reducing the spread of foliar diseases.

Use of fungicides may result in higher grain moisture. This can lead to increased costs associated with drying. So, it is important to investigate the best approach to reduce disease with minimum cost. To determine how to do so, we consider an optimal control problem.

The optimal control problem

To minimize the cost of implementing the controls, we assume that the control parameters ϕ, τ, θ in the control model are measurable functions of time and then formulate an appropriate optimal control functional that minimizes the cost of implementing the controls subject to the control model (12). For simplicity, we let $\phi = u_1(t),\ \tau = u_2(t),\ \theta = u_3(t)$. Therefore, the control scheme is said to be optimal if it minimizes the objective functional

$$\begin{aligned}\mathcal{J}(u_1,u_2,u_3) &= \int_0^{t_f} (A_1S(t) + A_2I(t) + A_3P(t) \\ &\quad + c_1u_1^2(t) + c_2u_2^2(t) + c_3u_3^2(t)\mathrm{d}t)\end{aligned} \tag{15}$$

subject to the control model, where t_f is the final time and the coefficients $A_1, A_2, A_3, c_1, c_2, c_3$ are balancing cost coefficients. The performance specification involves minimizing the number of infected maize plants and foliar pathogen, as well as the costs for applying the controls. We consider quadratic functions for measuring the control cost (Agusto 2009; Miller Neilan et al. 2010; Tchuenche et al. 2011).

The Pontryagins Maximum Principle (Pontryagin et al. 1986) introduces adjoint functions that enable us to combine the state system to the objective functional. This principle converts the problem of minimizing the objective functional subject to the state system into a problem of pointwise minimizing a Hamiltonian H, with respect to $u_1(t), u_2(t)$ and $u_3(t)$. The Hamiltonian for the objective functional and the state system is given by

$$\begin{aligned} H &= A_1S(t) + A_2I(t) + A_3P(t) + c_1u_1^2(t) + c_2u_2^2(t) \\ &\quad + c_3u_3^2(t) + \lambda_S(\Lambda - \beta S(t)P(t) - \mu_1S(t) - u_1(t)S(t) \\ &\quad + u_2(t)I(t)) + \lambda_C(u_1(t)S(t) - (1-\varepsilon)\beta C(t)P(t) \\ &\quad - \mu_3C(t)) + \lambda_I(\beta S(t)P(t) + (1-\varepsilon)\beta C(t)P(t) \\ &\quad - (\mu_2 + u_2(t))I) + \lambda_P(\sigma I(t) - (\xi + u_3(t))P(t)). \end{aligned} \tag{16}$$

Given an optimal control triple (u_1^*, u_2^*, u_3^*) together with corresponding states $(S_c^*, C_c^*, I_c^*, P_c^*)$ that minimizes $\mathcal{J}(u_1, u_2, u_3)$ over U, there exists adjoint variables λ_S, λ_C, λ_I and λ_P satisfying

$$\begin{aligned} \frac{\mathrm{d}\lambda_S}{\mathrm{d}t} &= -A_1 + \lambda_S(\beta P(t) + \mu_1 + u_1(t)) - \lambda_C u_1(t) - \lambda_I\beta P(t) \\ \frac{\mathrm{d}\lambda_C}{\mathrm{d}t} &= \lambda_C((1-\varepsilon)\beta P(t) + \mu_3) - \lambda_I(1-\varepsilon)\beta P(t) \\ \frac{\mathrm{d}\lambda_I}{\mathrm{d}t} &= -A_2 + \lambda_S u_2(t) + \lambda_I(\mu_2 + u_2(t)) - \lambda_P\sigma \\ \frac{\mathrm{d}\lambda_P}{\mathrm{d}t} &= -A_3 + \lambda_S\beta S(t) + \lambda_C(1-\varepsilon)\beta C(t) \\ &\quad - \lambda_I(\beta S(t) + (1-\varepsilon)\beta C(t)) + \lambda_P(\xi + u_3(t)). \end{aligned} \tag{17}$$

Together with transversality conditions:

$$\lambda_k(t_f) = 0 \quad \text{for } k = S, C, I, P. \tag{18}$$

The differential equations governing the adjoint variables were obtained by differentiating the Hamiltonian function with respect to the corresponding states as follows:

$$\frac{\mathrm{d}\lambda_k}{\mathrm{d}t} = -\mathrm{d}H/\mathrm{d}t. \tag{19}$$

Consider now the optimality conditions

$$0 = \frac{\mathrm{d}H}{\mathrm{d}u_1}, \quad 0 = \frac{\mathrm{d}H}{\mathrm{d}u_2}, \quad 0 = \frac{\mathrm{d}H}{\mathrm{d}u_3}. \tag{20}$$

By solving for u_1 in the optimality conditions and subsequently taking bounds into considerations, we obtain

$$u_1^* = \min(1, S(\lambda_S - \lambda_C)/2c_1). \qquad (21)$$

By a similar reasoning, we obtain

$$\begin{aligned} u_2^* &= \min(1, I(\lambda_I - \lambda_S)/2c_2), \\ u_3^* &= \min(1, P\lambda_P/2c_3). \end{aligned} \qquad (22)$$

These results demonstrate the existence of an optimal control triple (u_1^*, u_2^*, u_3^*) that can reduce the spread of foliar disease with minimum cost. Since the optimal control triple is parameter dependent, we use numerical simulations to determine their magnitudes for the period of the outbreak.

Numerical examples

The numerical solution of the optimal control triple is obtained with parameter values from Table 2 together with the following cost factors: $A_1 = 6.00$, $A_2 = 2.00$, $A_3 = 100.00$, $C_1 = 10.00$, $C_2 = 10.00$, $S(0) = 10.000$. We used the forward–backward algorithm of Lenhart and Workman (2007) and Miller Neilan et al. (2010) to obtain the optimal control triple that minimizes the cost functional. To achieve optimal cost control, the control measures are maximally applied at first but reduced toward harvest. This suggests that effective optimal controls should be applied from the onset of the outbreak. A numerical illustration of the number of infected maize plants in the presence or absence of control or with optimal control is presented in Figure 3.

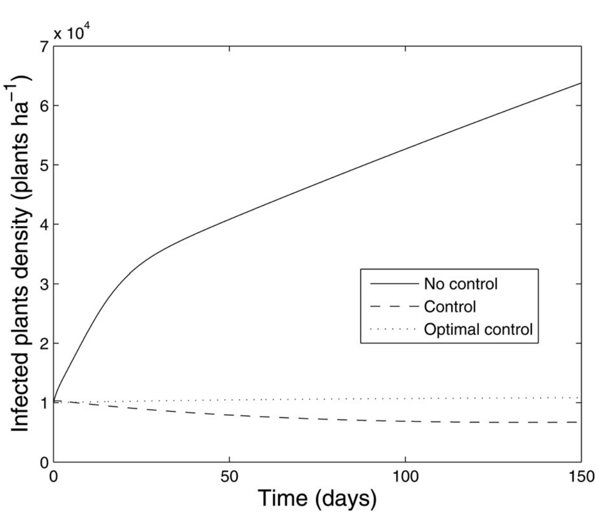

Figure 3. Plot showing the trajectories plants population density in the presence of control, no control and optimal control measure.

Discussion

This study investigated the impacts of foliar diseases on maize plant population dynamics and also evaluated how to reduce the spread of maize foliar diseases with minimum cost. For this, a deterministic differential equation model was developed to consider the importance of such epidemics.

First, we formulated an epidemiological model and analyses of this model revealed that foliar disease can spread without bound. Second, we incorporated control measures into the model and show how these can have significant effects in reducing the spread of the disease.

However, controlling diseases requires cost. Since most farmers need to optimize yields for profit, or for their own use, we investigated how to reduce the spread of diseases with minimum cost. For this we formulated an optimal control problem. Qualitative analyses suggest that introducing the control measure at the outset of the outbreak can reduce the spread of the diseases with minimum cost.

Disclosure statement

No potential conflict of interest was reported by the authors.

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